

Benha University Faculty of Engineering- Shoubra Eng. Mathematics & Physics Department Qualifying Courses (Mathematics)		Final Term Exam Date: January 16, 2016 Course: Linear Algebra EMM 402 Duration: 3 hours
• Answer All questions The exam consists of one page	• No. of questions: 5 Total Mark: 200	
Question 1		
(a) Determine the linearly independent and linearly dependent:	10	
(i) $u = (2, 3), v = (1, 2)$ (ii) $u = (2, 0, 3), v = (1, 2, 0), w = (3, 2, 3)$		
(b) If $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 0 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 & 2 \\ 0 & -2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \\ 3 & 2 & 0 \end{bmatrix}$	40	
Find, if possible, $A + B$, $A + B^t$, $A.B$, $A.B^t$, $A.C$, $ A $ and $ C $.		
Question 2		
(a) If $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{bmatrix}$.	40	
(i) Show that A is symmetric and find its eigenvalues and eigenvectors.		
(ii) Show that its eigenvectors are orthogonal.		
(b) Show that the eigenvalues of : $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ are real numbers, where a, b, c	10	
are real numbers.		
Question 3		
Write the following expressions in matrix form and determine the type:	30	
(a) $P = 4x^2 + 3y^2 + 2z^2 + 2xy - 2xz + yz$		
(b) $P = 4xy + 2xz - 2yz - 2x^2 - 3y^2 - 2z^2$		
(c) $P = 2xy - 6xz + 2yz + x^2 + 2y^2 + 3z^2$		
Question 4		
(a) Write the Hessian matrix of : $f(x, y, z) = xye^z + \cos y + x^3 \ln z$.	10	
(b) Find $f(A) = \frac{170A}{A^2+I}$ where $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 1 & 4 \end{bmatrix}$	20	
Question 5		
(a) Write the equations: $a_{11}x + a_{12}y + a_{13}z = b_1$, $a_{21}x + a_{22}y + a_{23}z = b_2$, $a_{31}x + a_{32}y + a_{33}z = b_3$ in matrix form and discuss the types of solutions.	20	
(b) Solve the linear system : $x - 2y + 2z = 2$, $2x + y - z = 4$, $2x + 2y - z = 6$, $3x - 2z = 2$	20	

Good Luck

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Answer

Answer of Question 1

$$(a)(i) au + bv = a(2, 3) + b(1, 2) = (2a + b, 3a + 2b) = 0$$

Then $2a + b = 0$, $3a + 2b = 0$. Then $a = b = 0$.

Then u and v are linearly independent.

$$(ii) au + bv + cw = a(2, 0, 3) + b(1, 2, 0) + c(3, 2, 3)$$

$$= (2a + b + 3c, 0a + 2b + 2c, 3a + 0b + 3c) = 0$$

Then $2a + b + 3c = 0$, $2b + 2c = 0$, $3a + 3c = 0$. Then $a = b = -c$.

Then u , v and w are linearly dependent.

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$$(b) A + B = \begin{bmatrix} 3 & -1 & 0 \\ 2 & -2 & -2 \end{bmatrix}$$

$A + B^T$, $A \cdot B$, $|A|$ are not exist.

$$A \cdot B^T = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ -3 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -8 & -6 \\ -2 & -3 \end{bmatrix}$$

$$A \cdot C = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \\ 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 2 & -4 \\ -7 & -6 & -4 \end{bmatrix}$$

$$|C| = 2 - 0 + 18 = 20$$

-----40-Marks

Answer of Question 2

$$(a)(i) \text{ Since } A^T = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{bmatrix} = A. \text{ Then, it is symmetric.}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\text{Then } \begin{vmatrix} 2-\lambda & 0 & -2 \\ 0 & 4-\lambda & 0 \\ -2 & 0 & 5-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda)(5-\lambda) - 4(4-\lambda) = 0$$

$$(4-\lambda)[(2-\lambda)(5-\lambda) - 4] = (4-\lambda)(\lambda-1)(\lambda-6) = 0$$

Then the eigenvalues are 1, 4, 6.

From the linear system of equations:

$$\begin{bmatrix} 2-\lambda & 0 & -2 \\ 0 & 4-\lambda & 0 \\ -2 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

For $\lambda = 1$, we get the system of equations:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0,$$

Then $x + 0y - 2z = 0$, $0x + 3y + 0z = 0$, $-2x + 0y + 4z = 0$

From the second equation $y = 0$.

From the first equation $x = 2z = \text{any number}$, put $z = 1$.

Then the corresponding eigenvector is $X_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = 4$, we get the system of equations: $\begin{bmatrix} -2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

Then $-2x + 0y - 2z = 0$, $0x + 0y + 0z = 0$, $-2x + 0y + z = 0$

From the first equation $y = \text{any number}$.

From the first and third equations $x = z = 0$. Putting $y = 1$.

Then the corresponding eigenvector is $X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = 6$, we get the system of equations:

$$\begin{bmatrix} -4 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0,$$

Then $-4x + 0y - 2z = 0$, $0x - 2y + 0z = 0$, $-2x + 0y - z = 0$

From the second equation $y = 0$.

From the first or third equations $z = -2x = \text{any number}$, putting $x = 1$.

Then the corresponding eigenvector is $X_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

The matrix of eigenvectors is $T = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$

(ii) We see that $X_1 \cdot X_2 = 0 = X_1 \cdot X_3 = X_2 \cdot X_3$

Then, the three eigenvectors are orthogonal.

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(b) Since $|A - \lambda I| = \begin{vmatrix} a - \lambda & c \\ c & b - \lambda \end{vmatrix} = (a - \lambda)(b - \lambda) - c^2 = 0$

Then, the characteristic equation is $\lambda^2 - (a + b)\lambda + ab - c^2 = 0$

The root of this equation are real if $(a + b)^2 \geq 4(ab - c^2)$

Then $a^2 + b^2 + 2ab \geq 4ab - 4c^2$

This inequality is true because $a^2 + b^2 - 2ab + 4c^2 \geq 0$

Or $(a - b)^2 + 4c^2 \geq 0$

Then the eigenvalues are real numbers

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Answer of Question 3

(a) $P = X^T A X = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 4 & 1 & -1 \\ 1 & 3 & \frac{1}{2} \\ -1 & \frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is positive definite.

(b) $P = X^T A X = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} -2 & 2 & 1 \\ 2 & -3 & -2 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is negative definite.

(C) $P = X^T A X = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 1 & -3 \\ 1 & 2 & 1 \\ -3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is indefinite.

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Answer of Question 4

$$(a) H = \begin{bmatrix} 6x \ln z & e^z & \frac{3x^2}{z} \\ e^z & -\cos y & xe^z \\ \frac{3x^2}{z} & xe^z & xye^z \end{bmatrix}$$

-----10-Marks

(b) The eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 4$.

$$\begin{aligned} p(\lambda) &= f(\lambda_1) \frac{(\lambda - \lambda_2)(\lambda - \lambda_3)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + f(\lambda_2) \frac{(\lambda - \lambda_1)(\lambda - \lambda_3)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + f(\lambda_3) \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \\ &= \frac{85}{6}(\lambda - 3)(\lambda - 4) - \frac{51}{2}(\lambda - 1)(\lambda - 4) + \frac{40}{3}(\lambda - 1)(\lambda - 3) \\ &= 2\lambda^2 - 25\lambda + 108 \end{aligned}$$

$$\text{Then } p(A) = 2A^2 - 25A + 108I = \begin{bmatrix} 85 & 4 & -30 \\ 0 & 51 & 0 \\ 0 & -11 & 40 \end{bmatrix}$$

-----20-Marks

Answer of Question 5

$$(a) \text{The matrix form } AX = B, \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

If $G = [A : B]$

This system has three types of solutions :

(i) One solution if $\text{rank } A = \text{rank } G = 3$.

(ii) Infinite number of solutions if $\text{rank } A = \text{rank } G < 3$.

(iii) No solution if $\text{rank } A \neq \text{rank } G$.

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$$(b) G = \left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 2 & 1 & -1 & 4 \\ 2 & 2 & -1 & 6 \\ 3 & 0 & -2 & 2 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 0 & 5 & -5 & 0 \\ 0 & 6 & -5 & 2 \\ 0 & 6 & -8 & -4 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

It has one solution: (2, 2, 2)

-----20-Marks

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Final Exam and ILOs

Course Title: Linear Algebra

Code: EMM 402

Questions	ILOs				
	Knowledge and Understanding		Intellectual Skills		Professional and Practical Skills
	2.1.1	2.1.2	2.2.3	2.2.7	2.3.2
Q1	√		√		
Q2			√	√	
Q3	√		√		
Q4		√			√
Q5	√	√		√	√

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